1.

$$f(x) = (2 - 5x)^{-2} \qquad |x| < \tfrac{2}{5}.$$

Find the binomial expansion of f(*x*), in ascending powers of *x*, as far as the term in *x*³, giving each coefficient as a simplified fraction.

(5)

1. Ensure there is a 1 to being with

$$f(x) = 2^{-2}(1 - \frac{5}{2}x)^{-2}$$

Use Binomial

$$(1+x)^n = 1 + \frac{nx}{1!} + \frac{n(n-1)x^2}{2!} + \cdots$$

$$f(x) = 2^{-2}(1 + (-2).\frac{-5}{2}x + \frac{(-2)(-3)}{2}(\frac{-5}{2}x)^2 + \frac{(-2)(-3)(-4)}{3 \times 2})(\frac{-5}{2}x)^3 \ldots.$$

$$f(x) = 2^{-2}\left(1 + 5x + \frac{75}{4}x^2 + \frac{125}{2}x^3\right)$$

$$f(x) = \left(\frac{1}{4} + \frac{5}{4}x + \frac{75}{16}x^2 + \frac{125}{8}x^3 \ldots\right)$$

2. The curve with equation

$$y = \frac{1}{3(1+2x)} \qquad x > -\frac{1}{2}$$

The region bounded by the lines $= -\frac{1}{4}, x = \frac{1}{2}$, the *x*-axis and the curve is shown shaded in Figure 1.
This region is rotated through 360 degrees about the *x*-axis.

(a) Use calculus to find the exact value of the volume of the solid generated.

(5)

(b) Figure 2 shows a paperweight with axis of symmetry *AB* where *AB* = 3 cm. *A* is a point on the top surface of the paperweight, and *B* is a point on the base of the paperweight. The paperweight is geometrically similar to the solid in part (a). Find the volume of this paperweight.

(2)

a) Use the volume of rotation around x-axis is $Vol = \pi \int y^2 dx$

$$Vol = \pi \int_{-1/4}^{1/2} \frac{1}{3^2(1+2x)^2} dx$$

$$Vol = \pi\frac{1}{9}\int_{-1/4}^{1/2}(1+2x)^{-2}dx$$

Using $\frac{1}{n+1}x^{n+1}$

$$Vol = \pi\frac{1}{9}\left[\frac{1}{-1}\cdot\frac{1}{2}(1+2x)^{-1}\right]$$

$$Vol = -\pi\frac{1}{18}[(1+2x)^{-1}]$$

$$= \left[-\frac{\pi}{18}(1+\frac{2}{2})^{-1}\right] - \left[-\frac{\pi}{18}(1-\frac{2}{4})^{-1}\right]$$

Note the first term has to be positive as it is a volume.

$$= -\frac{\pi}{36} + \frac{\pi}{9} = \frac{3\pi}{36} = \frac{\pi}{12}$$

b) The axis of the paper weight is exactly 4 times the above shape. Therefore the volume will be 4^3.

$$= 4^3\frac{\pi}{12} = \frac{64\pi}{12} = \frac{16\pi}{3}$$

3. A curve has parametric equations

$$x = 7\cos t - \cos 7t, \quad y = 7\sin t - \sin 7t \qquad \frac{\pi}{8} < t < \frac{\pi}{3}$$

(a) Find an expression for $\frac{dy}{dx}$ in terms of t. You need not simplify your answer.

(3)

(b) Find an equation of the normal to the curve at the point where $t = \frac{\pi}{6}$. Give your answer in its simplest exact form.

(6)

a) $\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx}$

$$\frac{dx}{dt} = -7\sin t + 7\sin 7t$$

$$\frac{dy}{dt} = 7\cos t - 7\cos 7t$$

Therefore

$$\frac{dy}{dx} = \frac{7\cos t - 7\cos 7t}{-7\sin t + 7\sin 7t}$$

For a normal $m = -\frac{dx}{dy}$ so

$$m = -\frac{dx}{dy} = -\frac{-7\sin\frac{\pi}{6} + 7\sin 7\frac{\pi}{6}}{7\cos\frac{\pi}{6} - 7\cos 7\frac{\pi}{6}}$$

$$m = -\frac{-\frac{7}{2} - \frac{7}{2}}{\frac{7\sqrt{3}}{2} + \frac{7\sqrt{3}}{2}} = \frac{14}{14\sqrt{3}} = \frac{1}{\sqrt{3}}$$

When $t = \frac{\pi}{6}$

$$y = 7\sin\frac{\pi}{6} - \sin\frac{7\pi}{6} = \frac{7}{2} + \frac{1}{2} = 4$$

$$x = 7\cos\frac{\pi}{6} - \cos\frac{7\pi}{6} = \frac{7\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = 4\sqrt{3}$$

$$y = \frac{1}{\sqrt{3}}x + c$$

Fill in x and y $\quad 4 = \frac{1}{\sqrt{3}}4\sqrt{3} + c \qquad c = 0$

Therefore $\quad y = \frac{1}{\sqrt{3}}x$

4. **a) Express $\frac{2x-1}{(x-1)(2x-3)}$ as partial fractions** **(3)**

b) Given that x≥2, find the general solution of the differential equation

$$(2x-3)(x-1)\frac{dy}{dx} = (2x-1)y$$

(5)

c) Hence find the particular solution of this differential equation that satisfies *y* = 10 at *x* = 2, giving your answer in the form *y* = f(*x*).

(4)

a) Let $\quad \frac{2x-1}{(x-1)(2x-3)} = \frac{A}{x-1} + \frac{B}{2x-3}$

Therefore $\quad \frac{2x-1}{(x-1)(2x-3)} = \frac{A(2x-3) + B(x-1)}{(x-1)(2x-3)}$

Top row only compare parts $\quad$ (x) $\quad 2 = 2A + B$ (1)

(nos) $\quad -1 = -3A - B$ (2)

Add the 1) and 2) $\quad 2 - 1 = 2A - 3A$

$1 = -A \quad and\ A = -1$

Sub back into 1) $\quad 2 = -2 + B \quad and\ B = 4$

b) Move all the x's to the right hand side (RHS) and the y's to the left hand side (LHS). $\quad \frac{1}{y}dy = \frac{2x-1}{(2x-3)(x-1)}dx$

Integrate both sides and substitute in part a) for RHS. $\quad \int \frac{1}{y}dy = \int -\frac{1}{(x-1)} + \frac{4}{(2x-3)}dx$

$$\ln|y| = -\ln|x-1| + 4.\frac{1}{2}\ln|2x-3| + \ln|k|$$

Using $aln|b| = \ln|b|^a$ $\quad$ $\ln y = -\ln|x-1| + \ln|2x-3|^2 + lnk$

Using $\ln|b| + \ln|a| = \ln|ba|$
And $\ln|b| - \ln|a| = \ln\left|\frac{b}{a}\right|$ $\quad$ $\ln y = \ln\frac{k(2x-3)^2}{(x-1)}$

Anti-logging $\quad$ $y = \frac{k(2x-3)^2}{(x-1)}$

c) When y=10, x=2. $\quad$ $10 = \frac{k(4-3)^2}{(2-1)}$ $\quad$ $k = 10$

Therefore $\quad$ $y = \frac{10(2x-3)^2}{(x-1)}$

5. A set of curves is given by the equation $sinx + cosy = 0.5$

(a) Use implicit differentiation to find an expression for $\frac{dy}{dx}$.

(2)

For –π < *x* < π and –π < *y* < π ,

(b) Find the coordinates of the points where $\frac{dy}{dx} = 0$.

(5)

a) Taking $\frac{d}{dx}$, don't forget to differentiate y with respect to itself and multiple by $\frac{dy}{dx}$. $\quad$ $\frac{d}{dx}(sinx + cosy) = 0$

$$cosx - siny\frac{dy}{dx} = 0$$

Rearrange $\quad$ $\frac{cosx}{siny} = \frac{dy}{dx}$

b) If $\frac{cosx}{siny} = \frac{dy}{dx} = 0$ $\quad$ Then $cosx = 0 \quad x = \frac{\pi}{2}\ or - \frac{\pi}{2}$

When $x = -\frac{\pi}{2}$ $\quad$ $sin - \frac{\pi}{2} + cosy = 0.5 \quad and\ \ cosy = 1.5\ which\ is\ not\ true.$

When $x = \frac{\pi}{2}$ $\quad$ $sin\frac{\pi}{2} + cosy = 0.5$

Therefore $\quad$ $cosy = -0.5 \quad y = cos^{-1}(-0.5) = \frac{2\pi}{3}\ or - \frac{2\pi}{3}$

Coordinates are $\quad$ $(x, y) = \left(\frac{\pi}{2}, \frac{2\pi}{3}\right)\ and\ \left(\frac{\pi}{2}, -\frac{2\pi}{3}\right)$

6. (a) Given that $y = 2^x$, and using the result $2^x = e^{xln2}$, or otherwise, show that $\frac{dy}{dx} = 2^x ln2$. **(2)**

(b) Find the gradient of the curve with equation $y = 2^{x^2}$ at the point with coordinates (2, 16). **(4)**

a) $y = e^{xln2}$

$\frac{dy}{dx} = ln2. e^{xln2} = ln2. 2^x$

b) Gradient is given by $\frac{dy}{dx} = m$

By comparison with part a)

$\frac{dy}{dx} = ln2. 2^{x^2}. 2x = 2xyln2$

When x=2, y=16

$\frac{dy}{dx} = 2.2.16. ln2 = 64ln2$

7. The point *A* has position vector a = 2i + 2j + k and the point *B* has position vector b = i + j – 4k, relative to an origin *O*.

(a) Find the position vector of the point *C*, with position vector c, given by c = a + b. **(1)**

(b) Show that *OACB* is a rectangle, and find its exact area. **(6)**

The diagonals of the rectangle, *AB* and *OC*, meet at the point *D*.
(c) Write down the position vector of the point *D*. **(1)**

(d) Find the size of the angle *ADC*. **(6)**

a) $c = a + b = 2i + 2j + k + i + j - 4k = 3i + 3j - 3k$

b) Prove 2 sets of 2 lengths are equal. OA.OB=0, BO.BC=0, AC.BC=0, AO.OC=0

Note these vectors can be either way round i.e OA or AO.

$|OA| = \sqrt{(2^2 + 2^2 + 1^2)} = \sqrt{9} = 3$

$|AC| = c - a = b = \sqrt{(1^2 + 1^2 + 4^2)} = \sqrt{18} = 3\sqrt{2}$

$|CB| = b - c = a = 3$

$OA. OB = (2i + 2j + k). (i + j - 4k) = 2 + 2 - 4 = 0$

$BO. BC = (-i - j + 4k). (2i + 2j + k) = -2 - 2 + 4 = 0$

$AC. BC = (i + j - 4k). (2i + 2j + k) = 2 + 2 = 4 = 0$

$AO. AC = (-2i - 2j - k). (i + j - 4k) = -2 - 2 + 4 = 0$

c) OD is half OC

$OD = \frac{3}{2}i + \frac{3}{2}j - \frac{3}{2}k$

d) Using $cos\theta = \frac{AD. DC}{|AD||DC|}$

$AD = d - a = \left(\frac{3}{2}i + \frac{3}{2}j - \frac{3}{2}k\right) - (2i + 2j + k)$

$AD = -\frac{1}{2}i - \frac{1}{2}j - \frac{1}{2}k$

$|AD| = \sqrt{\frac{1}{4} + \frac{1}{4} + \frac{1}{4}} = \sqrt{\frac{3}{4}}$

$$DC = d - c = \left(\frac{3}{2}i + \frac{3}{2}j - \frac{3}{2}k\right) - (3i + 3j - 3k)$$
$$DC = -\frac{3}{2}i - \frac{3}{2}j + \frac{3}{2}k$$

$$|DC| = \sqrt{\frac{9}{4} + \frac{9}{4} + \frac{9}{4}} = \sqrt{\frac{27}{4}}$$

$$cos\theta = \frac{AD.CD}{|AD||DC|} = \frac{\left(-\frac{1}{2}i - \frac{1}{2}j - \frac{1}{2}k\right).(+\frac{3}{2}i + \frac{3}{2}j - \frac{3}{2}k)}{\sqrt{\frac{3}{4}}.\sqrt{\frac{27}{4}}}$$

Note ACD must be greater than 90 degrees so ensure correct value.

$$cos\theta = -\frac{3/4}{9/3} = -\frac{1}{3} \qquad \theta = cos^{-1} - \left(\frac{1}{3}\right) \qquad \theta = 109.5°$$

8.

$$I = \int_0^5 e^{\sqrt{(3x+1)}}\, dx$$

(a) Given that $y = e^{\sqrt{(3x+1)}}$ complete the table with the values of *y* corresponding to *x* = 2, 3 and 4.

x	0	1	2	3	4	5
y	e^1	e^2	$e^{\sqrt{7}}$	$e^{\sqrt{10}}$	$e^{\sqrt{13}}$	e^4

(2)

(b) Use the trapezium rule, with all the values of *y* in the completed table, to obtain an estimate for the original integral *I*, giving your answer to 4 significant figures.

(3)

(c) Use the substitution *t* = √(3*x* + 1) to show that *I* may be expressed as $\int_a^b kte^t\, dt$, giving the values of *a*, *b* and *k*.

(5)

(d) Use integration by parts to evaluate this integral, and hence find the value of *I* correct to 4 significant figures, showing all the steps in your working.

(5)

a) Simply put x into the formula for y

b) Using

$$y \approx \frac{h}{2}\{y_0 + y_n + 2(y_1 + y_2 \ldots\ldots y_{n-1}\}$$

From table it is easy to see that h=1 i.e. 1-0 or 2-1 etc. Therefore

$$y \approx \frac{1}{2}\left\{e + e^4 + 2(e^2 + e^{\sqrt{7}} + e^{\sqrt{10}} + e^{\sqrt{13}}\right\} = 110.6(4.s.f)$$

c) If

$$t = \sqrt{3x + 1}$$

$t^2 = 3x + 1$

Differentiate w.r.t x $\quad 2tdt = 3dx \quad \frac{2}{3}tdt = dx$

When x=5 $\quad t = \sqrt{3 \times 5 + 1} = 4$

When x=0 $\quad t = \sqrt{3 \times 0 + 1} = 1$

Therefore $\quad \mathrm{I} = \int_0^5 \mathrm{e}^{\sqrt{(3x+1)}}\,\mathrm{dx} = \int_1^4 \mathrm{e}^{t}\frac{2}{3}\mathrm{tdt}$

Therefore $\quad b = 4, a = 1, k = \frac{2}{3}$

d) Let $\quad \mathrm{u} = \frac{2}{3}\mathrm{t} \quad \frac{du}{dt} = \frac{2}{3} \qquad \frac{dv}{dt} = e^t \qquad v = e^t$

Using $uv - \int v\frac{du}{dx}dx$

$$= \frac{2}{3}te^t - \int e^t \times \frac{2}{3}dt$$

$$= \left[\frac{2}{3}te^t - \frac{2}{3}e^t\right]_1^4$$

$$= \frac{(8e^4 - 2e^4) - (2e - 2e)}{3} = \frac{6e^4}{3} = 2e^4$$