1. The curve shown in Figure 1 has equation $y = e^x\sqrt{sinx}$ $\quad 0 \le x \le \pi$. The finite region *R* bounded by the curve and the *x*-axis is shown shaded in Figure 1.

(a) Complete the table below with the values of *y* corresponding to $x = \frac{\pi}{4}$ *and* $\frac{\pi}{2}$, giving your answers to 5 decimal places.

(2)

(b) Use the trapezium rule, with all the values in the completed table, to obtain an estimate for the area of the region *R*. Give your answer to 4 decimal places.

(4)

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
y	0	1.84432	4.81048	8.87207	0

a) Simply put the values of x into the formula in your calculator

PUT YOUR CALCULATOR IN RADIANS!

b) Using

$$y \approx \frac{h}{2}\{y_0 + y_n + 2(y_1 + y_2 \ldots\ldots y_{n-1}\}$$

$$h = \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

$$y \approx \frac{\pi}{8}\{0 + 0 + 2(1.84432 + 4.81048 + 8.87207\}$$

$y \approx 12.1948$ (4.d.p)

2. (a) Use the binomial theorem to expand

$$(8 - 3x)^{\frac{1}{3}}, \qquad |x| < \frac{8}{3}$$

in ascending powers of *x*, up to and including the term in x^3 , giving each term as a simplified fraction.

(5)

(b) Use your expansion, with a suitable value of *x*, to obtain an approximation to $\sqrt[3]{7.7}$. Give your answer to 7 decimal places.

(2)

a) Bring 8 out to ensure it starts with a 1.

$$(8 - 3x)^{\frac{1}{3}} = 8^{\frac{1}{3}}(1 - \frac{3}{8}x)^{\frac{1}{3}}$$

Using binomial expansion

$$(1 + x)^n = 1 + \frac{nx}{1!} + \frac{n(n-1)x^2}{2!} + \cdots$$

$$= 8^{\frac{1}{3}}(1 + \frac{1}{3}\left(-\frac{3}{8}x\right) + \frac{\frac{1}{3}\left(-\frac{2}{3}\right)}{2}(-\frac{3}{8}x)^2 + \frac{\frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{3 \times 2}(-\frac{3}{8}x)^3$$

$$= 2\left(1 - \frac{1}{8}x - \frac{1}{64}x^2 - \frac{5}{1536}x^3 \ldots.\right)$$

$$= 2 - \frac{1}{4}x - \frac{1}{32}x^2 - \frac{5}{768}x^3 \ldots..$$

b) Always compare with part a) so in this case $8 - 3x = 7.7 \quad x = 0.1$

Fill x=0.1 into expansion

$$= 2 - \frac{1}{4}(0.1) - \frac{1}{32}(0.1)^2 - \frac{5}{768}(0.1)^3 = 1.97468099 = 1.9746810\ (7.d.p)$$

3. The curve shown in Figure 2 has equation

$$\boldsymbol{y = \frac{1}{2x+1}}.$$

The finite region bounded by the curve, the *x*-axis and the lines *x = a* and *x = b* is shown shaded in Figure 2. This region is rotated through 360° about the *x*-axis to generate a solid of revolution. Find the volume of the solid generated. Express your answer as a single simplified fraction, in terms of *a* and *b*.

(5)

Using

$$Vol = \pi \int y^2 dx$$

$$Vol = \pi \int_a^b \frac{1}{(2x+1)^2} dx = \pi \int_a^b (2x+1)^{-2} dx$$

Using $\frac{1}{n+1}x^{n+1}$

$$= \pi \left[\frac{1}{-1}(2x+1)^{-1}.\frac{1}{2}\right]_a^b = -\frac{\pi}{2(2b+1)} + \frac{\pi}{2(2a+1)}$$

$$= \frac{\pi(-2a - 1 + 2b + 1)}{2(2b+1)(2a+1)} = \frac{\pi(b-a)}{(2b+1)(2a+1)}$$

4. (i) Find $\int \ln\left(\frac{x}{2}\right) dx$ **(4)**

(ii) Find the exact value of $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin^2 x \, dx$

(5)

a) To integrate lnx you integrate by parts.
Using

$$uv - \int v\frac{du}{dx}dx$$

$$u = \ln\left(\frac{x}{2}\right) \quad \frac{du}{dx} = \frac{2}{x}\left(\frac{1}{2}\right) = \frac{1}{x} \qquad \frac{dv}{dx} = 1 \quad v = x$$

$$= x\ln\frac{x}{2} - \int x.\frac{1}{x}dx = x\ln\frac{x}{2} - x + k$$

c) Always use the cos double angle formula to integrate cos^2x or sin^2x.

$$cos2x = cos^2x - sin^2x$$
$$cos2x = (1 - sin^2x) - sin^2x$$
$$cos2x = 1 - 2sin^2x$$
$$sin^2x = \frac{1}{2}(1 - cos2x)$$
$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} sin^2x\,dx = \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 1 - cos2xdx$$
$$\frac{1}{2}\left[x - \frac{1}{2}sin2x\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \frac{1}{2}\left(\frac{\pi}{2} - \frac{1}{2}sin\pi\right) - \frac{1}{2}\left(\frac{\pi}{4} - \frac{1}{2}sin\frac{\pi}{2}\right)$$
$$= \frac{\pi}{4} - \frac{\pi}{8} + \frac{1}{4} = \frac{\pi}{8} + \frac{1}{4}$$

5. A curve is described by the equation

$$\boldsymbol{x^3 - 4y^2 = 12xy.}$$

(a) Find the coordinates of the two points on the curve where *x* = –8.

(3)

(b) Find the gradient of the curve at each of these points.

(6)

a) Let x=-8

$$(-8)^3 - 4y^2 = 12(-8)y$$
$$(-8)^3 - 4y^2 = 12(-8)y$$

$$0 = 4y^2 - 96y + 512$$

Divide by 4

$$0 = y^2 - 24y + 128$$

$$0 = (y - 8)(y - 16)$$

Coordinates are $(-8,8)$ $(-8,16)$

b) Differentiate with respect to x but remember to differentiate y w.r.t itself and multiply by $\frac{dy}{dx}$. RHS is differentiation by parts

$$3x^2 - 8y\frac{dy}{dx} = 12x\frac{dy}{dx} + 12y$$
$$3x^2 - 12y = (12x + 8y)\frac{dy}{dx}$$

$$\frac{dy}{dx} = m$$

$$\frac{dy}{dx} = \frac{3x^2 - 12y}{12x + 8y}$$

At (-8,8)

$$\frac{dy}{dx} = \frac{3(-8)^2 - 12(8)}{12(-8) + 8(8)} = \frac{192 - 96}{-96 + 64} = -3$$

At (-8,16)

$$\frac{dy}{dx} = \frac{3(-8)^2 - 12(16)}{12(-8) + 8(16)} = \frac{192 - 96}{-96 + 64} = 0$$

6. The points *A* and *B* have position vectors 2i + 6j – k and 3i + 4j + k respectively. The line l_1 passes through the points *A* and *B*.

(a) Find the vector AB.

(2)

(b) Find a vector equation for the line.

(2)

A second line l_2 passes through the origin and is parallel to the vector i + k. The line meets the line l_1 at the point *C*.

(c) Find the acute angle between l_1 and l_2 .

(3)

(d) Find the position vector of the point *C*.

(4)

a) $AB = b - a = (3i + 4j + k) - (2i + 6j - k) = i - 2j + 2k$

b) A position on the line and direction vector. $r_1 = (2i + 6j - k) + t(i - 2j + 2k)$

c) Using $cos\theta = \frac{a.b}{|a||b|}$

$$cos\theta = \frac{(i - 2j + 2k).(i + k)}{\sqrt{(1 + 4 + 4)}\sqrt{(1 + 1)}} = \frac{3}{3\sqrt{2}}$$

$$\theta = \frac{\pi}{4}$$

At C find s and t

$r_2 = s(i + k)$
$r_1 = (2i + 6j - k) + t(i - 2j + 2k)$

Compare i, j and k

$(i)\quad s = 2 + t \quad (1)$
$(j)\quad 0 = 6 - 2t \quad (2)$
$(k)\quad s = -1 + 2t \quad (3)$

From (2)

$t = 3$
$s = 5$ from (1)
$s = -1 + 2(3)$

Therefore

$OC = 5i + 5k$

7. The curve *C* has parametric equations

$$x = \ln(t + 2), \quad y = \frac{1}{(t + 1)}, \quad t > -1$$

.

The finite region *R* between the curve *C* and the *x*-axis, bounded by the lines with equations
***x* = ln 2 and *x* = ln 4, is shown shaded in Figure 3.**

(a) Show that the area of *R* is given by the integral

(4)

$$\int_0^2 \frac{1}{(t + 1)(t + 2)} dt$$

(b) Hence find an exact value for this area.

(6)

(c) Find a cartesian equation of the curve *C*, in the form *y* = f(*x*).

(4)

(d) State the domain of values for *x* for this curve.

(1)

a) $Area = \int y\,dx$ $\qquad \frac{dx}{dt} = \frac{1}{t+2}$

When x=ln 4, t=2, when x=ln2, t=0. Therefore $\qquad Area = \int_0^2 \frac{1}{(t+1)(t+2)}dt$

b) When you have two brackets at the bottom you often use partial fractions so $\qquad \frac{1}{(t+1)(t+2)} = \frac{A}{t+1} + \frac{B}{t+2}$

Therefore $\qquad 1 = A(t+2) + B(t+1)$

Compare parts $\qquad (t)\;\; 0 = A + B \quad (1)$

$(nos)\;\; 1 = 2A + B \quad (2)$

Subtract (2) from(1) $\qquad -1 = -A \qquad A = 1, B = -1$

Integral is therefore $\qquad Area = \int_0^2 \frac{1}{t+1} - \frac{1}{t+2}dt = \left[\ln(t+1) - \ln(t+2)\right]_0^2 = ln\frac{t+1}{t+2}\Big]$

$= ln\frac{3}{4} - ln\frac{1}{2} = ln\frac{6}{4} = ln\frac{3}{2}$

c) Find t in terms of x $\qquad x = \ln(t+2)$

$e^x = t + 2$

$e^x - 2 = t$

Substitute t back into y $\qquad y = \frac{1}{(e^x - 1)}$

d) t>-1 therefore $\qquad x > \ln(1) \qquad x > 0\; x \in \mathbb{R}$

8. Liquid is pouring into a large vertical circular cylinder at a constant rate of 1600 cm^3 s^{-1} and is leaking out of a hole in the base, at a rate proportional to the square root of the height of the liquid already in the cylinder. The area of the circular cross section of the cylinder is 4000 cm^2.

(a) Show that at time *t* seconds, the height *h* cm of liquid in the cylinder satisfies the differential equation,

$$\frac{dh}{dt} = 0.4 - k\sqrt{h}$$

where *k* is a positive constant.

(3)

When $h = 25$, water is leaking out of the hole at 400 cm^3 s^{-1}.

(b) Show that $k = 0.02$

(1)

(c) Separate the variables of the differential equation

$$\frac{dh}{dt} = 0.4 - 0.02\sqrt{h},$$

to show that the time taken to fill the cylinder from empty to a height of 100 cm is given by

$$\int_0^{100} \frac{50}{20-\sqrt{h}}\,dh$$

(2)

Using the substitution $H = (20-x)^2$, or otherwise,
(d) find the exact value of

$$\int_0^{100} \frac{50}{20-\sqrt{h}}\,dh$$

(6)

(e) Hence find the time taken to fill the cylinder from empty to a height of 100 cm, giving your answer in minutes and seconds to the nearest second.

(1)

a) Vol of liquid = 1600 cm^3 s^{-1} $\quad \frac{dV}{dt} = 1600 - c\sqrt{h}$

Using $\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$ $\quad \frac{dV}{dh} = 4000$

Where $\frac{c}{4000} = k$ $\quad \frac{dh}{dt} = \frac{1600 - c\sqrt{h}}{4000} = 0.4 - k\sqrt{h}$

b) Note it is just rate OUT of hole so $\quad \frac{dh}{dt} = k\sqrt{h}$

$$\frac{400}{4000} = k\sqrt{25} \qquad k = 0.02$$

c) Take h to LHS and t to RHS $\quad \int \frac{1}{0.4 - 0.02\sqrt{h}}\,dh = \int dt$

Multiply top and bottom of LHS by 50 $\quad \int_0^{100} \frac{50}{20-\sqrt{h}}\,dh = t$

d) Using $H = (20-x)^2$ $\quad \int_0^{100} \frac{50}{20-(20-x)}\,dh = 50\int_0^{100} \frac{1}{x}\,dh$

Differentiate h in terms of x $\quad \frac{dh}{dx} = 2 \times (-1) \times (20-x) \qquad dh = (-40+2x)dx$

Find x when h=100 or 0.

$10 = 20 - x \quad x = 10$
$0 = 20 - x \quad x = 20$

$$50\int_{20}^{10} \frac{2x-40}{x}dx = 100\int_{20}^{10} 1 - \frac{20}{x}dx$$
$$= 100[x - 20lnx]_{20}^{10} = 100\big((10 - 20ln10) - 100(20 - 20ln20)\big)$$

$$= 100\left(-10 + 20ln\frac{20}{10}\right) = 2000ln2 - 1000$$

e) Calculate

$$= 2000ln2 - 1000 = 386.294\ secs = 6\ mins\ 26\ secs$$