1.

$$f(x) = (3+2x)^{-3}, \quad |x| < \frac{3}{2}$$

Find the binomial expansion of f (x) in ascending powers of x, up to and including the term in x^3. Give each coefficient as a simplified fraction.

(5)

1. Always ensure the brackets start with a 1 so bring the 3 outside.

$$f(x) = 3^{-3}(1+\frac{2x}{3})^{-3}$$

Using

$$(1+x)^n = 1 + \frac{nx}{1!} + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} +$$

$$f(x) = 3^{-3}(1 - 3\left(\frac{2x}{3}\right) + \frac{-3\times-4}{2}(\frac{2x}{3})^2 + \frac{-3\times-4\times-5}{3\times2}(\frac{2x}{3})^3 + \cdots)$$

$$f(x) = 3^{-3}(1 - 2x + \frac{24}{9}x^2 - \frac{80}{27}x^3 + \cdots)$$

$$f(x) = \frac{1}{27} - \frac{2}{27}x + \frac{8}{81}x^2 - \frac{80}{729}x^3 + \cdots)$$

2. Use the substitution $u = 2^x$ to find the exact value of

$$\int_0^1 \frac{2^x}{(2^x+1)^2}dx$$

(6)

2. If $u = 2^x \quad lnu = xln2$

$$\frac{1}{ln2} \times \frac{1}{u}du = dx$$

$u = 2^x \quad \frac{1}{u}du = ln2\, dx$

When $x = 1 \quad u = 2^1 = 2$

$x = 0 \quad u = 2^0 = 1$

$$\int_0^1 \frac{2^x}{(2^x+1)^2}dx = \frac{1}{ln2}\int_1^2 \frac{u}{(u+1)^2}\frac{1}{u}du = \frac{1}{ln2}\int_1^2 \frac{1}{(u+1)^2}du$$

Using

$$\frac{1}{n+1}x^{n+1}$$

$$= \frac{1}{ln2}\int_1^2 (u+1)^{-2}\,du = \frac{1}{ln2}(-1)(u+1)^{-1}]_1^2$$

$$= -\frac{1}{ln2}\times\frac{1}{3} + \frac{1}{ln2}\times\frac{1}{2} = \frac{1}{ln2}\left(\frac{3-2}{6}\right) = \frac{1}{6ln2}$$

3. (a) Find $\int xcos2xdx$.

(4)

(b) Hence, using the identity $cos2x = 2cos^2x - 1$, deduce $\int\, x\, cos^2\, x\, dx$.

(3)

3 a) Using Integration by parts

$$= uv - \int v\frac{du}{dx}dx$$

$$u = x \quad \frac{du}{dx} = 1 \qquad \frac{dv}{dx} = cos2x \quad v = \frac{1}{2}sin2x$$

$$= \frac{x}{2}sin2x - \frac{1}{2}\int sin2xdx$$

$$= \frac{x}{2}sin2x - \frac{1}{2}.\frac{1}{2} - cos2x \; + C$$

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$= \frac{x}{2} sin2x + \frac{1}{4} cos2x + C$

b) Rearranges $cos2x = 2cos^2x - 1$

$\frac{1}{2}(cos2x + 1) = cos^2x$

Therefore $\int x\, cos^2\, x\; dx = \frac{1}{2}\int xcos2x + x\, dx$

$= \frac{1}{2}\left(\frac{x}{2} sin2x + \frac{1}{4} cos2x + \frac{x^2}{2}\right) + C$

$= \frac{x}{4} sin2x + \frac{1}{8} cos2x + \frac{x^2}{4} + C$

4.

$$\frac{2(4x^2+1)}{(2x+1)(2x-1)} \equiv A + \frac{B}{(2x+1)} + \frac{C}{(2x-1)}$$

(a) Find the values of the constants *A*, *B* and *C*.

(4)

(b) Hence show that the exact value of $\int_1^2 \frac{2(4x^2+1)}{(2x+1)(2x-1)} dx$, is $2 + lnk$ giving the value of the constant *k*.

(6)

4 a) Find common denominators of LHS $\equiv \frac{A(4x^2+1) + B(2x-1) + C(2x+1)}{(2x+1)(2x-1)}$

Compare x^2 terms $8 \equiv 4A \quad A = 2$

Let $x=\frac{1}{2}$ $2(1+1) \equiv 2C \quad C = 2$

Let $x=-\frac{1}{2}$ $2(1+1) \equiv -2B \quad B = -2$

$$\frac{2(4x^2+1)}{(2x+1)(2x-1)} \equiv 2 - \frac{2}{(2x+1)} + \frac{2}{(2x-1)}$$

b)

$$\int_1^2 \frac{2(4x^2+1)}{(2x+1)(2x-1)} dx = \int_1^2 2 - \frac{1}{(2x+1)} + \frac{1}{(2x-1)} dx$$

$$= 2x - \frac{2}{2} ln(2x+1) + \frac{2}{2} ln(2x-1)\Big]_1^2$$

$$= 2x + ln\frac{(2x-1)}{(2x+1)}\Big]_1^2 = 4 + ln\frac{3}{5} - 2 - ln\frac{1}{3}$$

$$= 2 + ln\frac{9}{5}$$

5.

The line l_1 has equation $\boldsymbol{r} = \begin{pmatrix}1\\0\\-1\end{pmatrix} + \lambda\begin{pmatrix}1\\1\\0\end{pmatrix}$

The line l_2 has equation $\boldsymbol{r} = \begin{pmatrix}1\\3\\6\end{pmatrix} + \mu\begin{pmatrix}2\\1\\-1\end{pmatrix}$

(a) Show that l_1 and l_2 do not meet.

(4)

The point *A* is on l_1 where λ= 1, and the point *B* is on l_2 where μ= 2.

(b) Find the cosine of the acute angle between *AB* and l_1.

(6)

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5 a) Set up three equations and prove there is no values that work for all three.

$$\boldsymbol{r} = \begin{pmatrix}1\\0\\-1\end{pmatrix} + \lambda\begin{pmatrix}1\\1\\0\end{pmatrix}$$
$$\boldsymbol{r} = \begin{pmatrix}1\\3\\6\end{pmatrix} + \mu\begin{pmatrix}2\\1\\-1\end{pmatrix}$$

First equation set up $\quad 1 + \lambda = 1 + 2\mu \qquad \lambda = 2\mu$

$\lambda = 3 + \mu \qquad from\ above \quad 2\mu = 3 + \mu \qquad \mu = 3 \qquad and\ \lambda = 6$

Third equation check $\quad -1 = 6 - \mu \qquad \mu = 7 \neq 3$

Therefore they do not meet.

b) When λ=1

$$\boldsymbol{A} = \begin{pmatrix}1\\0\\-1\end{pmatrix} + 1\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}2\\1\\-1\end{pmatrix}$$

When μ=2

$$\boldsymbol{B} = \begin{pmatrix}1\\3\\6\end{pmatrix} + 2\begin{pmatrix}2\\1\\-1\end{pmatrix} = \begin{pmatrix}5\\5\\4\end{pmatrix}$$

Find AB=b-a $\quad AB = 3i + 4j + 5k$

Using

$$cos\theta = \frac{a.b}{|a||b|}$$

$$cos\theta = \frac{(3i + 4j + 5k).(i + j)}{\sqrt{(3^2 + 4^2 + 5^2)}.\sqrt{1^2 + 1^2}} = \frac{7}{\sqrt{50}\sqrt{2}} = \frac{7}{10}$$

6. A curve has parametric equations

$$\boldsymbol{x = tan^2 t \qquad y = sint \qquad 0 < t < \frac{\pi}{2}}$$

(a) Find an expression for $\frac{dy}{dx}$ in terms of *t*. You need not simplify your answer.

(3)

(b) Find an equation of the tangent to the curve at the point where $t = \frac{\pi}{4}$.
Give your answer in the form $= ax + b$, where *a* and *b* are constants to be determined.

(5)

(c) Find a cartesian equation of the curve in the form $y^2 = f(x)$.

(4)

6a)

$$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx}$$

$$\frac{dy}{dt} = cost \quad x^{\frac{1}{2}} = tant \quad \frac{1}{2}x^{-\frac{1}{2}}dx = sec^2tdt \quad \frac{dx}{dt} = 2sec^2t\sqrt{x}$$

$$\frac{dx}{dt} = 2sec^2t\ tant$$

$$\frac{dy}{dx} = \frac{cost}{2sec^2t\ tant} = \frac{cos^3t}{2\tan t} = \frac{cos^4t}{2sint}$$

b) Tangent is when

$$\frac{dy}{dx} = m$$

$$\frac{dy}{dx} = m = \frac{cos^4\frac{\pi}{4}}{2sin\frac{\pi}{4}} = \frac{\sqrt{2}}{8} \qquad y = \frac{\sqrt{2}}{8}x + C$$

When t=$\frac{\pi}{4}$ $\quad x = tan^2\left(\frac{\pi}{4}\right) = 1$

When t=$\frac{\pi}{4}$ $\quad y = sint = \frac{\sqrt{2}}{2}$

Plug in to find C

$$\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{8} + C \quad C = \frac{4\sqrt{2} - 2}{8} = C = \frac{3\sqrt{2}}{8}$$

$$y = \frac{\sqrt{2}}{8}x + \frac{3\sqrt{2}}{8}$$

c) $x = tan^2t \qquad y = sint$

$$y^2 = sin^2t \quad x = tan^2t = \frac{sin^2t}{cos^2t} = \frac{sin^2t}{1 - sin^2t} = \frac{y^2}{1 - y^2}$$

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$$x = \frac{y^2}{1-y^2} \qquad x(1-y^2) = y^2 \quad x - xy^2 = y^2$$
$$x = y^2 + xy^2 \qquad x = y^2(1+x)$$
$$\frac{x}{1+x} = y^2$$

7. Figure 1 shows part of the curve with equation $y = \sqrt{(tanx)}$. The finite region *R*, which is bounded by the curve, the *x*-axis and the line $x = \frac{\pi}{4}$, is shown shaded in Figure 1.

(a) Given that $y = \sqrt{(tanx)}$, complete the table with the values of *y* corresponding to $x = \frac{\pi}{16}, \frac{\pi}{8}, \frac{3\pi}{16}$, giving your answers to 5 decimal places.

X	0	$\frac{\pi}{16}$	$\frac{\pi}{8}$	$\frac{3\pi}{16}$	$\frac{\pi}{4}$
y	0	0.44600	0.64359	0.81742	1

(3)

(b) Use the trapezium rule with all the values of *y* in the completed table to obtain an estimate for the area of the shaded region *R*, giving your answer to 4 decimal places.

(4)

The region *R* is rotated through 2π radians around the *x*-axis to generate a solid of revolution.

(c) Use integration to find an exact value for the volume of the solid generated.

(4)

a) Simply put the values into the calculator. Ensure calculator is in radians

b) Using

$$y \approx \frac{h}{2}\{y_0 + y_n + 2(y_1 + y_2 \ldots\ldots y_{n-1})\}$$

h=$\frac{\pi}{16}$

$$y \approx \frac{\pi}{32}\{0 + 1 + 2(0.44600 + 0.64359 + 0.81742)\}$$
$$y = 0.4726\ (4.d.p)$$

c) Using

$$Vol = \pi \int y^2 dx$$

$$Vol = \pi \int_0^{\frac{\pi}{4}} tanx\ dx = \ln \sec x]_0^{\frac{\pi}{4}} = \pi lnsec\frac{\pi}{4} - \pi lnsec0$$
$$= -\pi lncos\frac{\pi}{4} + \pi lncos0 = -\pi ln\frac{\sqrt{2}}{2} + \pi ln1 = \pi ln\frac{2}{\sqrt{2}}$$

8. A population growth is modelled by the differential equation where *P* is the population, *t* is the time measured in days and *k* is a positive constant.

$$\frac{dP}{dt} = kP$$

Given that the initial population is P_0,
(a) solve the differential equation, giving *P* in terms of P_0, *k* and *t*.

(4)

Given also that *k* = 2.5,
(b) find the time taken, to the nearest minute, for the population to reach $2P_0$.

(3)

In an improved model the differential equation is given as

$$\frac{dP}{dt} = \lambda P cos\lambda t$$

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where *P* is the population, *t* is the time measured in days and is a positive constant.

Given, again, that the initial population is P_0 and that time is measured in days,
(c) solve the second differential equation, giving *P* in terms of P_0, and *t*.

(4)

Given also that λ= 2.5,
(d) find the time taken, to the nearest minute, for the population to reach $2P_0$ for the first time, using the improved model.

(3)

a) Separate terms on left and right.

$$\int \frac{1}{P} dP = k \int dt$$

$$lnP = kt + C$$

When t=0 P=P_0

$$lnP_0 = C$$
$$lnP = kt + lnP_0$$
$$lnP - lnP_0 = kt$$
$$ln\frac{P}{P_0} = kt \qquad \frac{P}{P_o} = e^{kt} \qquad P = P_0e^{kt}$$

b)

$$2P_0 = P_0e^{2.5t}$$

Taking natural logs of both sides

$$ln2 = 2.5t \quad t = \frac{1}{2.5}ln2 = 0.2772588\ days$$
$$= 0.2772588 \times 24 = 6.654212\ hrs$$
$$= 6hrs\ and\ 39\ mins$$

c)

$$\frac{dP}{dt} = \lambda P cos\lambda t$$

$$\int \frac{1}{P} dP = \lambda \int cos\lambda t dt$$

$$lnP = \frac{\lambda}{\lambda} sin\lambda t + C$$

When t=0 P=P_0
Therefore

$$lnP_0 = C$$
$$lnP = sin\lambda t + lnP_0$$
$$ln\frac{P}{P_0} = sin\lambda t$$
$$P = P_0e^{sin\lambda t}$$

d)

$$2P_0 = P_0e^{sin2.5t}$$

Taking natural logs

$$ln2 = sin2.5t \qquad t = \frac{1}{2.5}sin^{-1}(ln2) = 0.306338477\ days$$

$$t = 0.306338477 = 441.127\ mins = 7hrs\ and\ 21\ mins$$

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