1.

$$f(x) = \frac{1}{\sqrt{(4+x)}}, \quad |x| < 4$$

Find the binomial expansion of f (x) in ascending powers of x, up to and including the term in x^3. Give each coefficient as a simplified fraction.

(6)

1.

Always ensure the brackets start with a 1 so bring the 4 outside.

$$f(x) = (4+x)^{-\frac{1}{2}}$$
$$f(x) = 4^{-\frac{1}{2}}(1+\frac{x}{4})^{-\frac{1}{2}}$$

Using

$$(1+x)^n = 1 + \frac{nx}{1!} + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} +$$
$$f(x) = 4^{-\frac{1}{2}}(1 - \frac{1}{2}\left(\frac{x}{4}\right) + \frac{-\frac{1}{2}.-\frac{3}{2}}{2}(\frac{x}{4})^2 + \frac{-\frac{1}{2}.-\frac{3}{2}.-\frac{5}{2}}{3\times 2}(\frac{x}{4})^3 + \ )$$
$$= \frac{1}{2}(1 - \frac{x}{8} + \frac{3}{8}(\frac{x}{4})^2 - \frac{5}{16}(\frac{x}{4})^3 +..\ )$$
$$= \frac{1}{2}\left(1 - \frac{x}{8} + \frac{3}{128}x^2 - \frac{5}{1024}x^3 +..\ \right)$$
$$= \frac{1}{2} - \frac{x}{16} + \frac{3}{256}x^2 - \frac{5}{2048}x^3 +..$$

2. Figure 1 shows the finite region R bounded by the x-axis, the y-axis and the curve with equation $y = 3\cos\left(\frac{x}{3}\right), 0 \le x \le \frac{3\pi}{2}$

The table shows corresponding values of x and y for $y = 3\cos\left(\frac{x}{3}\right)$

x	**0**	$\frac{3\pi}{8}$	$\frac{3\pi}{4}$	$\frac{9\pi}{8}$	$\frac{3\pi}{2}$
y	**3**	**2.77164**	**2.12132**		**0**

(a) Complete the table above giving the missing value of y to 5 decimal places.

(1)

(b) Using the trapezium rule, with all the values of y from the completed table, find an approximation for the area of R, giving your answer to 3 decimal places.

(4)

(c) Use integration to find the exact area of R.

(3)

2a) Simply put the value into the formula. Calculator in radians.

$$y = 3\cos\left(\frac{9\pi}{8}.\frac{1}{3}\right) = 1.14805\ (5.d.p)$$

b) Using $y \approx \frac{h}{2}\{y_0 + y_n + 2(y_1 + y_2 \ldots\ldots y_{n-1})\}$

$h = \frac{3\pi}{8} - 0 = \frac{3\pi}{8}$ $\qquad y \approx \frac{3\pi}{16}\{3 + 0 + 2(2.77164 + 2.12132 + 1.14805)\} = 8.884\ (3.d.p)$

c) $\boldsymbol{y = 3\cos\left(\frac{x}{3}\right)}$ $\qquad R = 3\int_0^{\frac{3\pi}{2}} \cos\left(\frac{x}{3}\right)dx = 3\left[\frac{1}{1/3} sin\frac{x}{3}\right]_0^{\frac{3\pi}{2}} = 9sin\frac{\pi}{2} = 9$

3.

$$\boldsymbol{f(x) = \frac{4 - 2x}{(2x+1)(x+1)(x+3)} = \frac{A}{2x+1} + \frac{B}{x+1} + \frac{C}{x+3}}$$

(a) Find the values of the constants *A*, *B* and *C*.

(4)

(b) (i) Hence find $\int f(x)\, dx$.

(3)

(ii) Find $\int_0^2 f(x)dx$ in the form 1n *k*, where *k* is a constant.

(3)

3a) Find common denominator for RHS $\qquad = \frac{A(x+1)(x+3) + B(2x+1)(x+3) + C(2x+1)(x+1)}{(2x+1)(x+1)(x+3)}$

Take $x = -\frac{1}{2}$ to get rid of B & C $\qquad 4 + 2\left(\frac{1}{2}\right) = A\left(\frac{1}{2}\right)\left(\frac{5}{2}\right) \qquad A = 4$

Take $x = -1$ to get rid of A and C $\qquad 4 + 2 = B(-1)(2) \qquad B = -3$

Take $x = -3$to get rid of A and B $\qquad 4 + 6 = C(-5)(-2) \qquad C = 1$

b) $\qquad \int f(x)dx = \int \frac{4}{2x+1} - \frac{3}{x+1} + \frac{1}{x+3}dx$

$= 4\left(\frac{1}{2}\right)\ln(2x+1) - 3\ln(x+1) + \ln(x+3) + lnk$

$= \ln(2x+1)^2 - \ln(x+1)^3 + \ln(x+3) + lnk$

$= \ln\frac{k(2x+1)^2(x+3)}{(x+1)^3}$

c) $\int_0^2 f(x)dx$ $\qquad = \ln\frac{(4+1)^2(2+3)}{(2+1)^3} - \ln\frac{(0+1)^2(0+3)}{(0+1)^3}$

$= \ln\frac{(5)^3}{27} - \ln 3 = \ln\frac{125}{81}$

4. The curve *C* has the equation

$$\boldsymbol{ye^{-2x} = 2x + y^2}$$

(a) Find $\frac{dy}{dx}$ in terms of *x* and *y*.

(5)

The point *P* on *C* has coordinates (0, 1).

(b) Find the equation of the normal to *C* at *P*, giving your answer in the form *ax* + *by* + *c* = 0, where *a*, *b* and *c* are integers.

(4)

4a) Use differentiation by parts and differentiate y with respect to itself and multiple y by $\frac{dy}{dx}$

$$\frac{dy}{dx}e^{-2x} - 2e^{-2x}y = 2 + 2y\frac{dy}{dx}$$

$$\frac{dy}{dx}e^{-2x} - 2y\frac{dy}{dx} = 2e^{-2x}y + 2$$

$$\frac{dy}{dx}(e^{-2x} - 2y) = 2e^{-2x}y + 2$$

$$\frac{dy}{dx} = \frac{2(e^{-2x}y + 1)}{(e^{-2x} - 2y)}$$

B) For normal $\frac{dy}{dx} = -\frac{1}{m}$. Find $\frac{dy}{dx}$ at P(0,1)

$$\frac{dy}{dx} = \frac{2(e^0 + 1)}{(e^0 - 2)} = -4 \qquad m = \frac{1}{4}$$

Therefore

$$y = \frac{1}{4}x + c$$

Put P(0,1)

$$1 = 0 + c \qquad c = 1 \qquad y = \frac{1}{4}x + 1$$

Multiple by 4

$$4y = x + 4 \qquad 0 = x - 4y + 4$$

5. Figure 2 shows a sketch of the curve with parametric equations

$$x = 2\cos 2t, \; y = 6\sin t, \quad 0 \le t \le \frac{\pi}{2}$$

(a) Find the gradient of the curve at the point where $t = \frac{\pi}{3}$.

(4)

(b) Find a cartesian equation of the curve in the form

$$y = f(x), -k \le x \le k,$$

stating the value of the constant *k*.

(4)

(c) Write down the range of f (*x*).

(2)

5a) Using

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$\frac{dy}{dt} = 6cost$$

$\frac{dx}{dt} = -4sin2t$

Therefore $\frac{dy}{dx} = \frac{6cost}{-4sin2t} = -\frac{3cost}{2sin2t} = -\frac{3}{2sint}$

At $t = \frac{\pi}{3}$ $\frac{dy}{dx} = -\frac{3}{2sin\frac{\pi}{3}} = -\frac{\sqrt{3}}{2}$

b) $x = 2\,cos\,2t\,, y = 6\,sin\,t$ $\frac{y}{6} = sint$ $cos2t = \frac{x}{2} = cos^2t - sin^2t = (1 - sin^2t) - sin^2t$

$\frac{x}{2} = 1 - 2sin^2t = 1 - 2(\frac{y}{6})^2$

$\frac{x}{2} = 1 - 2(\frac{y}{6})^2$ $x = 2 - \frac{y^2}{9}$ $9x = 18 - y^2$

Put in limits in t to find limits in x. $y^2 = 18 - 9x$ $y = \sqrt{18 - 9x} = 3\sqrt{(2 - x)}$

$x = 2\,cos\,2t\;\;for\;\;0 \le t \le \frac{\pi}{2}$ $-2 \le x \le 2$

c)For $0 \le t \le \frac{\pi}{2}$ find y $y = 6\,sin\,t$ $0 \le y \le 6$ $0 \le f(x) \le 6$

6. (a) Find $\int \sqrt{(5 - x)}\, dx$.

(2)

Figure 3 shows a sketch of the curve with equation

$$y = (x - 1)\sqrt{(5 - x)}, \quad 1 \le x \le 5$$

(b) (i) Using integration by parts, or otherwise, find

$$\int (x - 1)\sqrt{(5 - x)} dx$$

(4)

(ii) Hence find

$$\int_1^5 (x - 1)\sqrt{(5 - x)}\; dx$$

(2)

6 a) Using $\frac{1}{n+1}x^{n+1}$

$\int \sqrt{5 - x}\,dx = \int (5 - x)^{\frac{1}{2}}\,dx = -\frac{1}{3/2}(5 - x)^{\frac{3}{2}} + C$

$= -\frac{2}{3}(5 - x)^{\frac{3}{2}} + C$

b) (i) Using $uv - \int v\frac{du}{dx}dx$

$\int (x - 1)\sqrt{(5 - x)}dx$

$u = x - 1$ $\frac{du}{dx} = 1$ $\frac{dv}{dx} = \sqrt{5 - x}$ $v = -\frac{2}{3}(5 - x)^{\frac{3}{2}}$

$= -\frac{2}{3}(x - 1)\,(5 - x)^{\frac{3}{2}} + \frac{2}{3}\int (5 - x)^{\frac{3}{2}}\,dx$

$$= -\frac{2}{3}(x-1)(5-x)^{\frac{3}{2}} - \frac{2}{3}\cdot\frac{2}{5}(5-x)^{\frac{5}{2}} + K$$

$$= -\frac{2}{3}(x-1)(5-x)^{\frac{3}{2}} - \frac{4}{15}(5-x)^{\frac{5}{2}} + K$$

(ii) $\int_1^5 (x-1)\sqrt{(5-x)}\ dx$

$$= (-\frac{2}{3}(4)(0)^{\frac{3}{2}} - \frac{4}{15}\left(0)^{\frac{5}{2}}\right) - (-\frac{2}{3}(0)(4)^{\frac{3}{2}} - \frac{4}{15}(4)^{\frac{5}{2}}$$

$$= 0 - 0 - 0 + \frac{4}{15}(2)^5 = \frac{128}{15}$$

7. Relative to a fixed origin *O*, the point *A* has position vector (8i + 13j – 2k), the point *B* has position vector (10i + 14j – 4k), and the point *C* has position vector (9i + 9j + 6k).

The line *l* passes through the points *A* and *B*.

(a) Find a vector equation for the line *l*.

(3)

(b) Find *CB*.

(2)

(c) Find the size of the acute angle between the line segment *CB* and the line *l*, giving your answer in degrees to 1 decimal place.

(3)

(d) Find the shortest distance from the point *C* to the line *l*.

(3)

The point *X* lies on *l*. Given that the vector *CX* is perpendicular to *l*,

(e) find the area of the triangle *CXB*, giving your answer to 3 significant figures.

(3)

7 a) The vector equation of line uses any point on the line and then the direction vector AB.

$$AB = b - a = 10i + 14j - 4k - 8i - 13j + 2k = 2i + j - 2k$$

$$r = (8i + 13j - 2k) + k(2i + j - 2k)$$

b) $CB = b - c$

$$\boldsymbol{CB} = (10\text{i} + 14\text{j} - 4\text{k}) - (9\text{i} + 9\text{j} + 6\text{k}) = \text{i} + 5\text{j} - 10\text{k}$$

$$CB = \sqrt{1^2 + 5^2 + 10^2} = \sqrt{126} = 11.2$$

c) Use

$$cos\theta = \frac{a.b}{|a||b|}$$

$$cos\theta = \frac{(2\text{i} + \text{j} - 2\text{k}).(i + 5j - 10k)}{\sqrt{(2^2 + 1 + 2^2)}\sqrt{(1^2 + 5^2 + 10^2)}} = \frac{2 + 5 + 20}{\sqrt{9\sqrt{126}}} = \frac{27}{3\sqrt{126}}$$

$$= \frac{3}{\sqrt{14}}$$

$$\theta = cos^{-1}\left(\frac{3}{\sqrt{14}}\right) = 36.699 = 36.7°$$

d) Recognise the right angled triangle. They are leading through this!

$$\frac{d}{\sqrt{126}} = sin\theta \qquad d = 6.708\ (3.d.p)$$

e) The same triangle where BX is base and XC is height. Use Pythagoras to find BX

$$BX^2 = 126 - d^2 = 81 \quad BX = 9$$

Using

$$Area = \frac{1}{2} base \times height$$

$$Area = \frac{1}{2} \times 9 \times 6.708 = 30.19 = 30.2\ (1.d.p)$$

8 (a) Using the identity $\cos 2\theta = 1 - 2\sin^2\theta$, find $\int \sin^2\theta \, d\theta$.

(2)

Figure 4 shows part of the curve *C* with parametric equations

$$x = \tan\theta, \quad y = 2\sin 2\theta, \quad 0 \le \theta < \frac{\pi}{2}$$

The finite shaded region *S* shown in Figure 4 is bounded by *C*, the line $x = \frac{1}{\sqrt{3}}$ and the *x*-axis. This shaded region is rotated through 2π radians about the *x*-axis to form a solid of revolution.

(b) Show that the volume of the solid of revolution formed is given by the integral

$$k\int_0^{\frac{\pi}{6}} \sin^2\theta \, d\theta$$

where *k* is a constant.

(5)

(c) Hence find the exact value for this volume, giving your answer in the form $p\pi^2 + q\pi\sqrt{3}$, where *p* and *q* are constants.

(3)

8a), find $\int \sin^2\theta \, d\theta$.

$$\cos 2\theta = 1 - 2\sin^2\theta$$

Therefore

$$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$$

$$\int \sin^2\theta \, d\theta = \int \frac{1}{2}(1 - \cos 2\theta)d\theta = \frac{1}{2}\left(\theta - \frac{1}{2}\sin 2\theta\right) + C$$

b) Volume of revolution is $\pi \int y^2 dx$

$$x = \tan\theta \qquad \frac{dx}{d\theta} = \sec^2\theta \qquad dx = \sec^2\theta d\theta$$

Using $\sin 2x = 2\sin x\cos x$

$$Vol = 4\pi \int \sin^2 2\theta \sec^2\theta d\theta$$

$$Vol = 16\pi \int \sin^2\theta\cos^2\theta \sec^2\theta d\theta = 16\pi \int \sin^2\theta \, d\theta$$

When $x = \frac{1}{\sqrt{3}}$ $\theta = \tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6}$

When $x = 0$ $\theta = \tan^{-1}0 = 0$

$$Vol = 16\pi \int_0^{\frac{\pi}{6}} \sin^2\theta \, d\theta$$

c)

$$= 16\pi \int_0^{\frac{\pi}{6}} \sin^2\theta \, d\theta = \frac{16\pi}{2}\left(\theta - \frac{1}{2}\sin 2\theta\right)\Big]_0^{\frac{\pi}{6}}$$

$$= 8\pi\left(\frac{\pi}{6}-\frac{1}{2}sin\frac{2\pi}{6}\right)\Big] \quad - 8\pi\left(0-\frac{1}{2}sin0\right)\Big]_0^{\frac{\pi}{6}} = 8\pi\left(\frac{\pi}{6}-\frac{1}{2}\times\frac{\sqrt{3}}{2}\right)$$

$$= \frac{4}{3}\pi^2 - 2\pi\sqrt{3}$$